# StatML: Introduction to Large Language Models

History, Transformer, Inference, and Post-training

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# The plan for today

- Part 1: A (Very) Brief History of Probabilistic Language Modelling
  - Word embeddings, sequence-to-sequence, GPT
- Part 2: Decoder Transformer Architecture
  - Tokenizer
  - Multi-headed Attention
  - Positional Encodings
- Part 3: Architecture Variations, Inference, Post-Training
  - Architecture choices (shapes, normalizations, activations)
  - Inference: prefill vs decode, attention variants (MQA, GQA, MLA)
  - Post-training: SFT vs RLHF

# Part 1: A (Very) Brief History of Probabilistic Language Modelling

# (Probabilistic) Language Models

- Probability of word sequence $w_{1:T}$

$$p(w_{1:T}) = \prod_{t=1}^{T} p(w_t \mid w_{1:t-1})$$

where $w_{i:j} = \{w_i, w_{i+1}, \ldots, w_{j-1}, w_j\}$.

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- N-grams: Markov assumption

$$p(w_t \mid w_{1:t-1}) \approx p(w_t \mid w_{t-N+1:t-1})$$

  but it destroys long-range context.

# Early Word Embeddings **(Bengio et al., 2003)**

- Learn an embedding matrix $C \in \mathbb{R}^{V \times D}$ and a predictor $f_\Theta$

$$p(w_t \mid w_{1:t-1}) \approx f_\Theta\big(c_{I(w_{t-1})}, \ldots, c_{I(w_{t-N})}\big),$$

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- Discrete words have no notion of "small change"; vectors do.
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## Training and early limitations

- **Approach:** Next token prediction from $N$ previous; optimize cross-entropy over $V$ words.
- **The N-gram Era:** Pre-GPU compute and shallow architectures:
  - *"[N]eural probabilistic language models remain far less widely used than N-gram models due to their notoriously long training times..."* (Mnih & Teh, 2012)
- Overcoming required new hardware (GPUs) and architectures (RNNs, Transformers).

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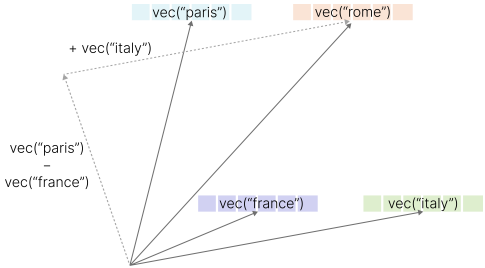
- **Emergent Structure:** Simple dot products learned complex geometry.
- Analogies became linear math:
  $\text{vec(King)} - \text{vec(Man)} + \text{vec(Woman)} \approx \text{vec(Queen)}$




Linear structure emerges in the embedding space.

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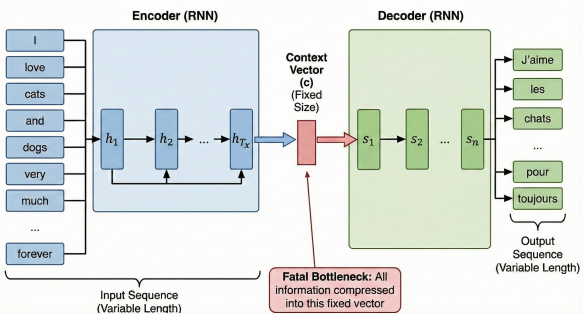
## **Idea 3:** Learn embeddings for variable length contexts

- **seq2seq:** Encoder–Decoder to learn a variable-length input $X$ to a variable-length output $Y$.



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- Keep encoder states for *all* tokens and *let the decoder decide* what is relevant.
- Dynamic $c_i$ as a weighted sum of all token hidden states $c_i = \sum_{j=1}^{T_x} \alpha_{ij} h_j$

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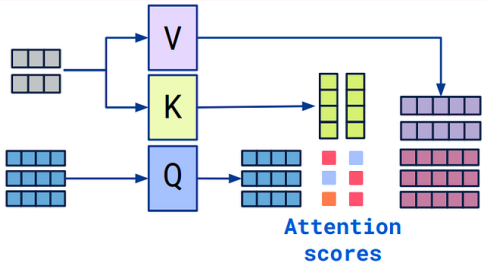
**Translating to modern QKV (Cross-Attention):**

- **Query ($q_i$):** What the decoder wants ← Previous decoder state $s_{i-1}$
- **Key/Value ($k_j, v_j$):** What the encoder holds ← Encoder states $h_j$



# The Transformer **(Vaswani et al., 2017)**

- **Sequential representations:** RNNs decoders are inherently sequential ($h_t$ depends on $h_{t-1}$). This precludes parallelization within a sequence, strictly limiting training scale.
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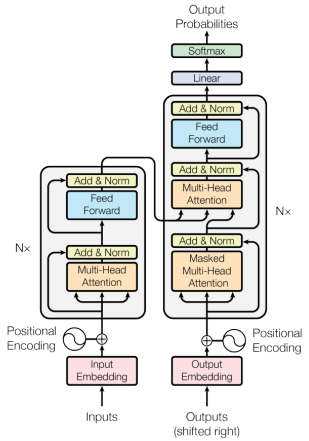
**Architectural changes:**

- **Self-Attention:** Remove RNN. Words attend to other words in the *same* sequence to build representations.
- **Positional Encoding:** Attention processes everything at once, must add positions.
- **Multi-Head:** Multiple attention representations in parallel.



# Generative Pre-Training **(Radford et al., 2019 - GPT-2)**

- **Massive data:** Use massive unlabeled web corpora (instead of labeled).
- **Decoder Only:** Train decoder-only Transformer using *next-word prediction*.

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Maximize the probability of the next token $w_t$ given all previous context $w_{<t}$:

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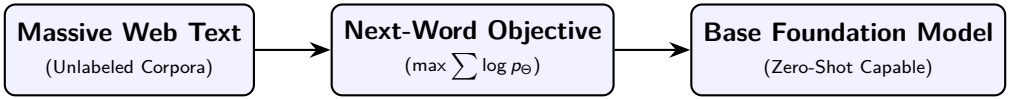

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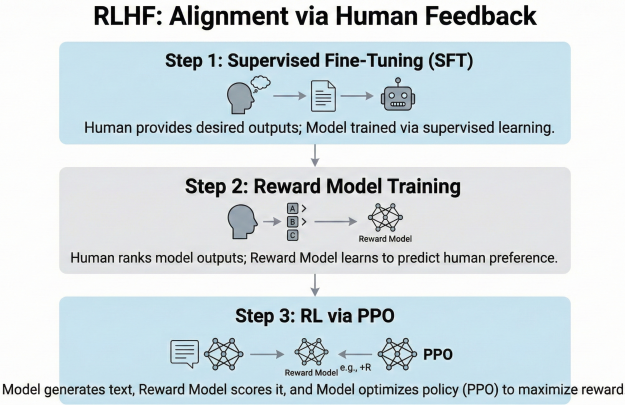
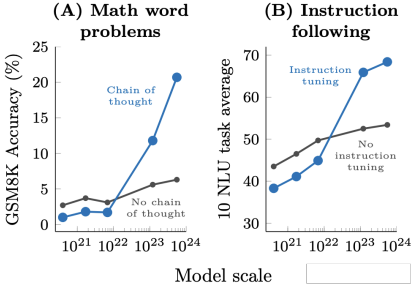
## Three steps:

1. **SFT:** Supervised fine-tuning on high-quality human demonstrations.
2. **Reward Model:** Train a "critic" network on human rankings to predict preference.
3. **RL (PPO):** Optimize the model's policy to maximize that reward score.



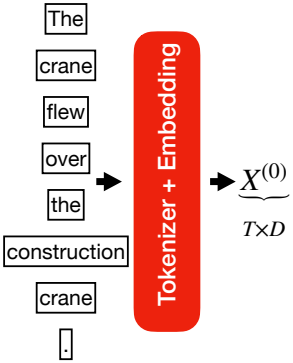
# Emergent Abilities of LLMs **(Wei et al., 2022)**

- **Emergent ability:** Not present at small scale.
- **Phase Shift:** A few-shot prompted tasks sharply works well for large models.
- **Measuring Scale:** Training FLOPS.
- **Unpredictability:** Cannot be predicted looking at scaling plots.



# Part 2: Decoder Transformer Architecture

# GPT2-like Decoder Transformer

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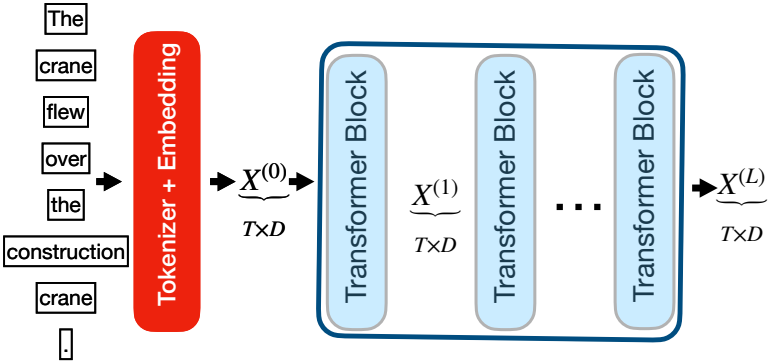


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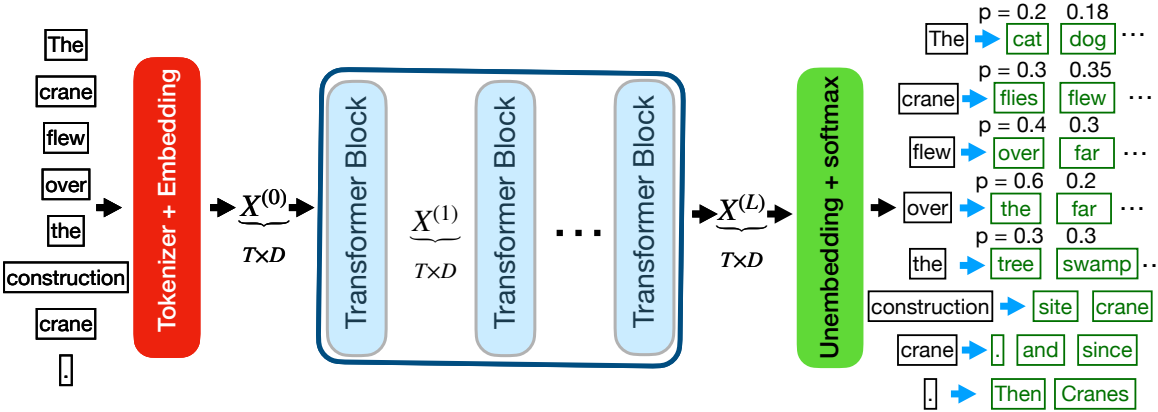


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Unembedding + Softmax : $\mathbb{R}^{T \times D} \to \mathbb{R}^{T \times |V|}$ probability over token vocabulary

# Step 1: Tokenizer + Embedding

# Step 1: Tokenization, vocabulary vs. length tradeoff

## Question

What is the right granularity to discretize text at?

- Raw text must be converted into discrete units $\rightarrow$ *tokenization*
- The set of possible tokens defines the model's **vocabulary**

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## The tradeoff

**Vocabulary size** $V$ vs **Sequence length** $T$ (and attention cost $\sim T^2$).

# Step 1: Byte Pair Encoding (BPE): Learning the vocabulary

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## BPE toy example)

**1. Corpus with counts:**

```
"hug": 10, "pug": 5, "pun": 12, "bun": 4,
"hugs": 5
```

**2. Split into characters:**

```
("h", "u", "g"): 10
("p", "u", "g"): 5
("p", "u", "n"): 12 ...
```

**3. Find the most frequent pair:**

The pair `("u", "g")` appears $10 + 5 + 5 = 20$ times.

**4. Merge Rule #1:** `"u" + "g"` $\rightarrow$ `"ug"`

**Updated Corpus:**

```
("h", "ug"): 10
("p", "ug"): 5
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```

## BPE Algorithm

Initialize vocab $V$ with all base bytes/chars.

1. Count adjacent pairs $(a, b)$.
2. Pick the most frequent pair.
3. Create new token $c \leftarrow ab$.
4. Repeat $M$ times.

Final vocab size: $|V_{\text{initial}}| + M$.

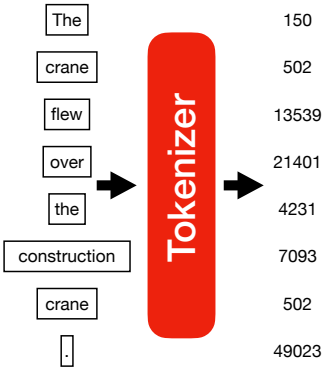
# Step 1: After tokenization: IDs $\rightarrow$ vectors

- Tokenizer outputs integer IDs: $\mathrm{tok}_t \in \{1, \ldots, V\}$.
- Show https://tiktokenizer.vercel.app/ or the Jupyter Notebook
- Convert IDs to vectors via an embedding table $E \in \mathbb{R}^{V \times d}$:

$$\text{one-hot}(\mathrm{tok}_t)^\top E \;=\; E_{\mathrm{tok}_t} \in \mathbb{R}^d \quad \text{(implemented as a lookup)}.$$

- Add positional information (absolute or relative) before attention.
- Transformer maps embeddings $\rightarrow$ logits in $\mathbb{R}^V$ for next-token prediction.

# Step 1: Tokenizer + Embedding

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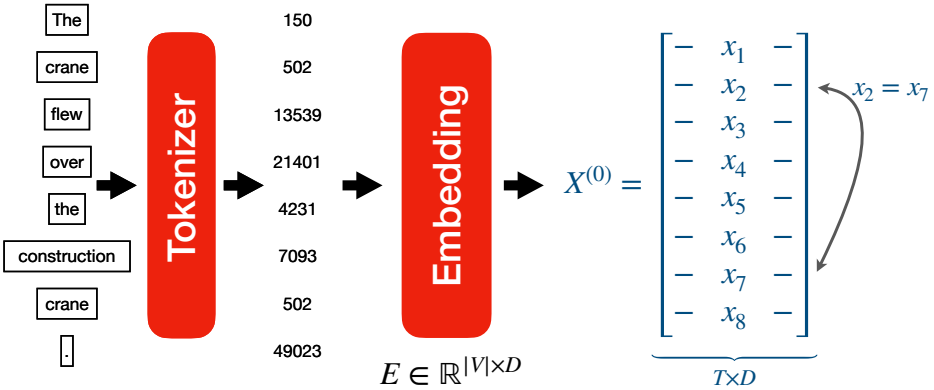


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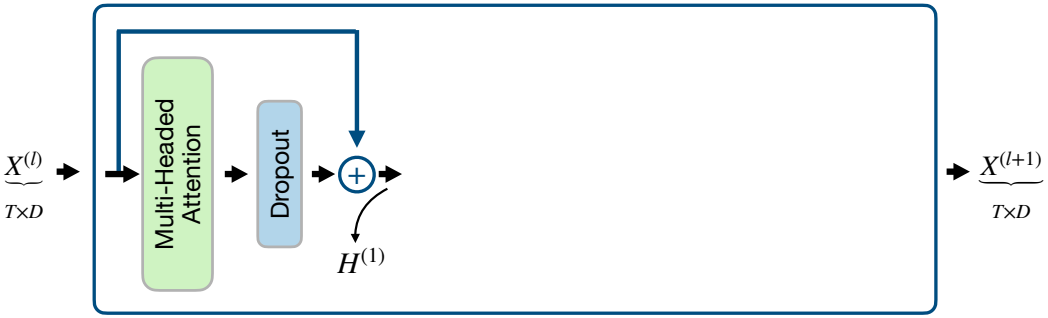
# Step 2: Transformer Decoder Block

# Step 2: Transformer (Decoder) Block




$\text{MHA} : \mathbb{R}^{T\times D} \to \mathbb{R}^{T\times D}$ Multi-Headed Attention

$\text{Dropout} : \mathbb{R}^{T\times D} \to \mathbb{R}^{T\times D}$ Set each entry to zero with prob. p

$\text{Residual connection} \oplus : \mathbb{R}^{T\times D} \to \mathbb{R}^{T\times D}$ Addition with the input

$$H^{(1)} = \text{Dropout}_p\left(\text{MHA}(X^{(l)})\right) + X^{(l)} \in \mathbb{R}^{T\times D}$$

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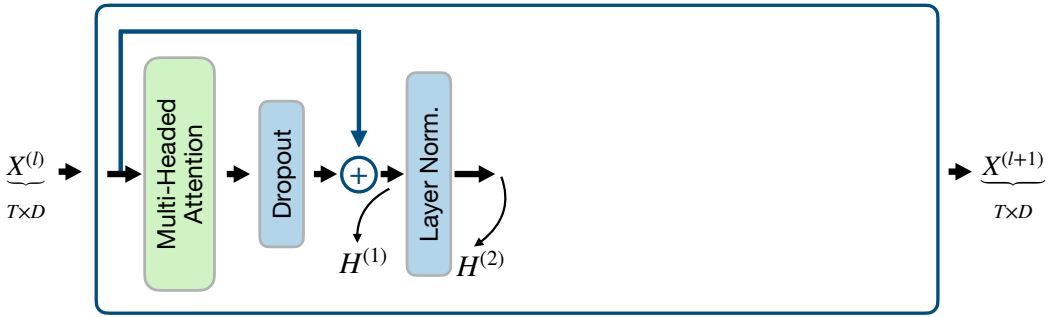



$\mathrm{LN} : \mathbb{R}^{T\times D} \to \mathbb{R}^{T\times D}$

Layer Norm. forces each token embedding (not the sequence) to have zero mean and st. dev 1

$$\underbrace{H^{(2)}}_{T\times D} = \mathrm{LN}_\varepsilon\left(H^{(1)}\right) = \left(\frac{H^{(1)} - \mathbf{1}^\top \mu_{[1:T]}}{\sqrt{\varepsilon + \mathbf{1}^\top \sigma^2_{[1:T]}}}\right) \cdot \gamma + \beta$$

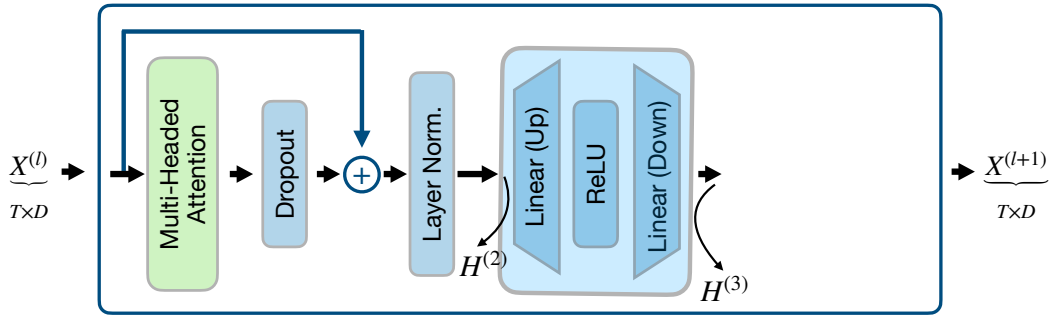
means: $\mu_{[1:T]} \in \mathbb{R}^T$

variances: $\sigma^2_{[1:T]} \in \mathbb{R}^T$

$\varepsilon = 10^{-6}$

learnable parameters: $\gamma, \beta$

# Step 2: Transformer (Decoder) Block




$\text{FFN} : \mathbb{R}^{T\times D} \to \mathbb{R}^{T\times D}$ Feed Forward Network

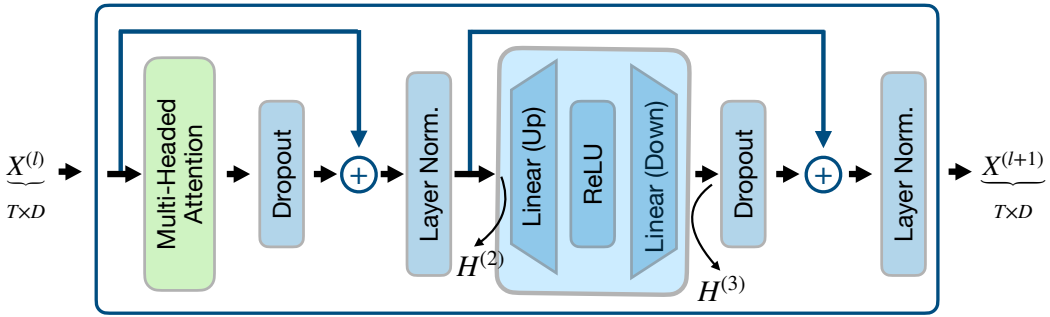
$$H^{(3)} = \text{FFN}(H^{(2)}) = \text{ReLU}(H^{(2)}W_1 + \mathbf{1}b_1^T)W_2 + \mathbf{1}b_2^T$$

$W_1 \in \mathbb{R}^{D\times 4D}$, $b_1 \in \mathbb{R}^{4D}$

$\text{ReLU}(x) = \max\{0, x\}$

$W_2 \in \mathbb{R}^{4D\times D}$, $b_2 \in \mathbb{R}^{D}$

# Step 2: Transformer (Decoder) Block




$$X^{(l+1)} = \mathrm{LN}_{\varepsilon}\left(\mathrm{Dropout}_{p}\left(H^{(3)}\right) + H^{(2)}\right) \in \mathbb{R}^{T \times D}$$

# Step 2: Multi-Headed Attention (MHA) with causal mask

$\underbrace{X^{(l)}}_{T \times D}$

The 3 aspects of every token

- Query $\left(Q \in \mathbb{R}^{T \times D}\right)$: “What information I look for?”
- Key $\left(K \in \mathbb{R}^{T \times D}\right)$: “What information do I offer?”
- Value $\left(V \in \mathbb{R}^{T \times D}\right)$: “What is my meaning?”

$\underbrace{\mathrm{MHA}(X^{(l)})}_{T \times D}$

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$$\underbrace{X^{(l)}}_{T\times D}$$

$$X^{(l)}W_K \to K = \underbrace{\begin{bmatrix} - & k_1 & - \\ - & k_2 & - \\ & \vdots & \\ - & k_T & - \end{bmatrix}}_{T\times D}$$

$$X^{(l)}W_Q \to Q = \underbrace{\begin{bmatrix} - & q_1 & - \\ - & q_2 & - \\ & \vdots & \\ - & q_T & - \end{bmatrix}}_{T\times D}$$

$$X^{(l)}W_V \to V = \underbrace{\begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_T & - \end{bmatrix}}_{T\times D}$$

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Parameters $W_Q, W_K, W_V \in \mathbb{R}^{D\times D}$

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Parameters $W_Q, W_K, W_V \in \mathbb{R}^{D\times D}$

Multiple roles of every token

- "Grammar", "Semantic", "Punctuation"
- Split to $H$ different $d = D/H$-dim. "heads"
- $\left\{Q^{(h)}, K^{(h)}, V^{(h)}\right\}_{h=1}^{H}$, where $Q^{(h)}, K^{(h)}, V^{(h)} \in \mathbb{R}^{T\times d}$

$\underbrace{\mathrm{MHA}(X^{(l)})}_{T\times D}$

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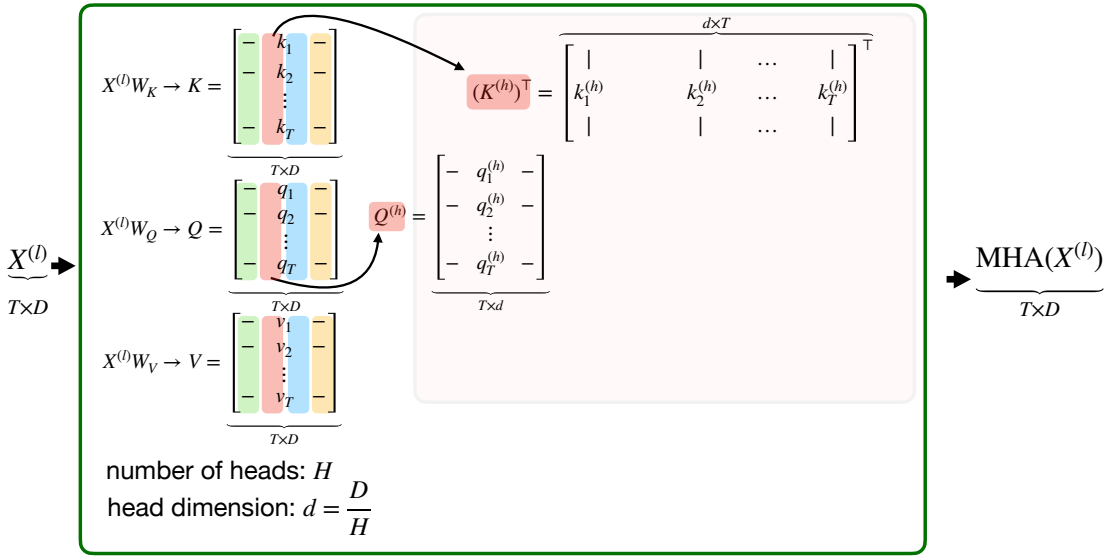

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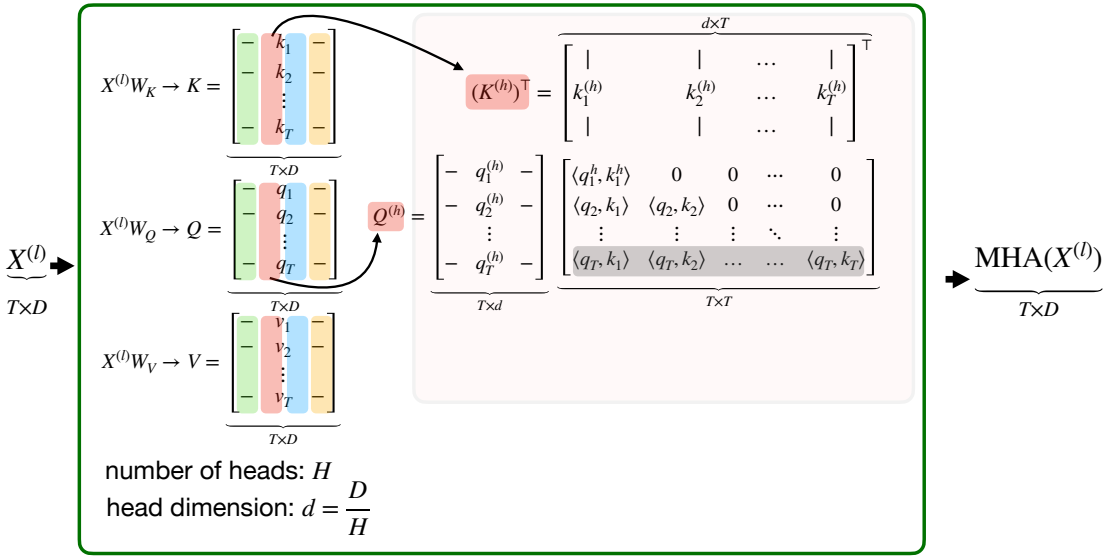

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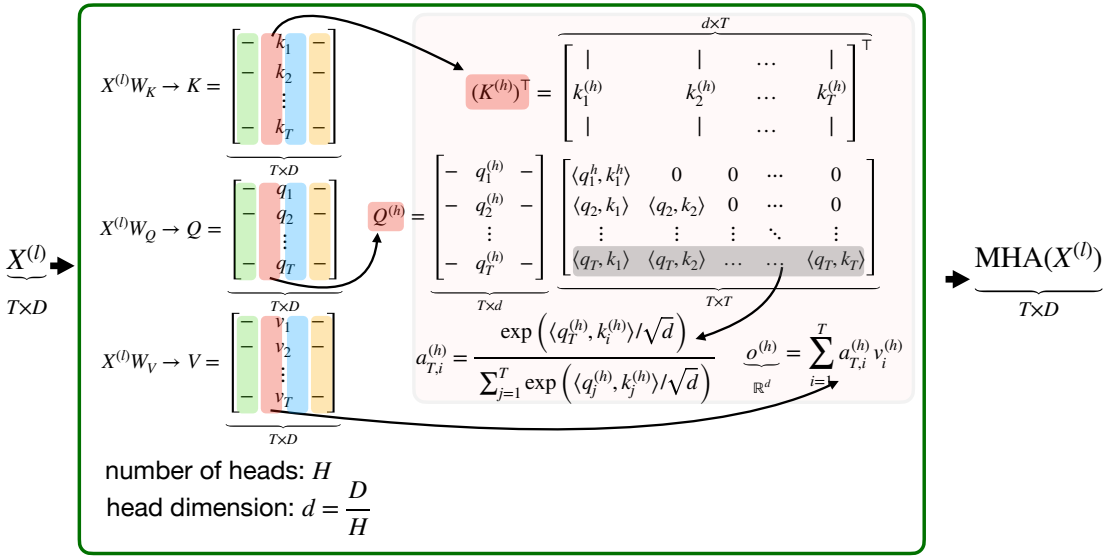

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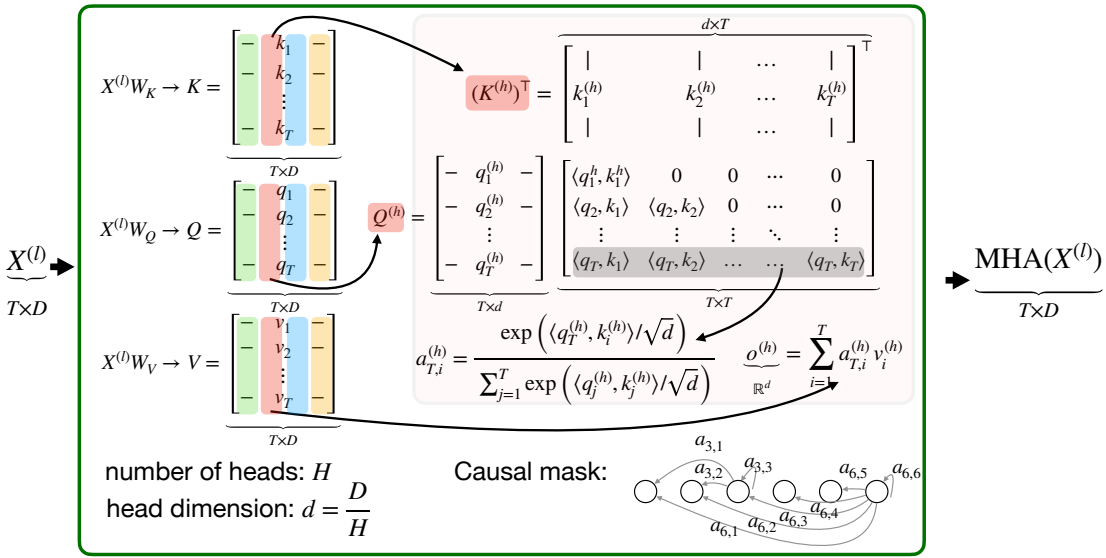

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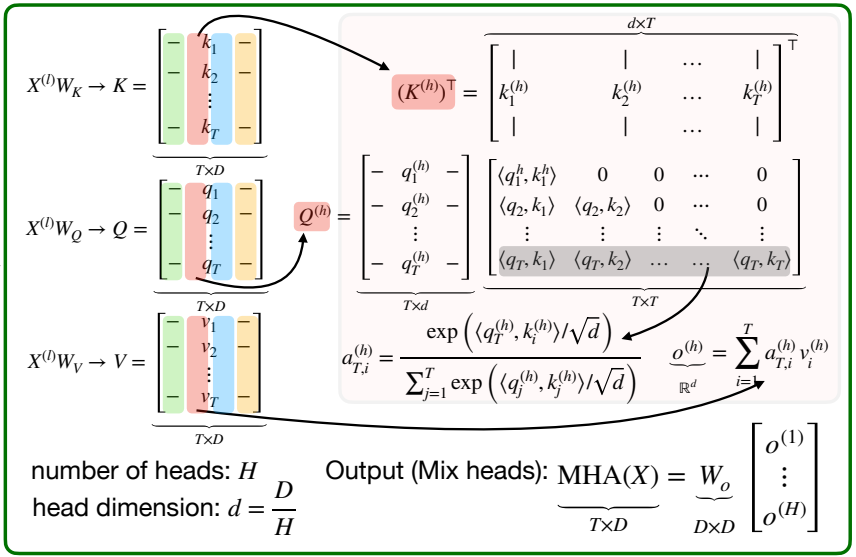

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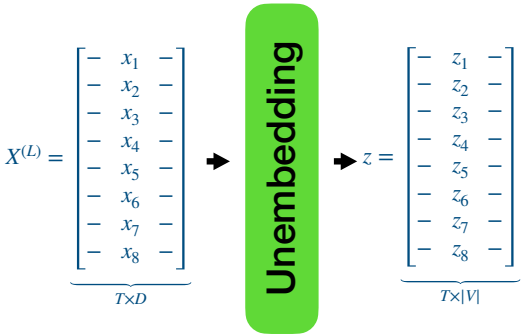


$$\underbrace{X^{(L)}}_{T\times D} \cdot \underbrace{U}_{D\times|V|} + b = \underbrace{Z}_{T\times|V|}$$

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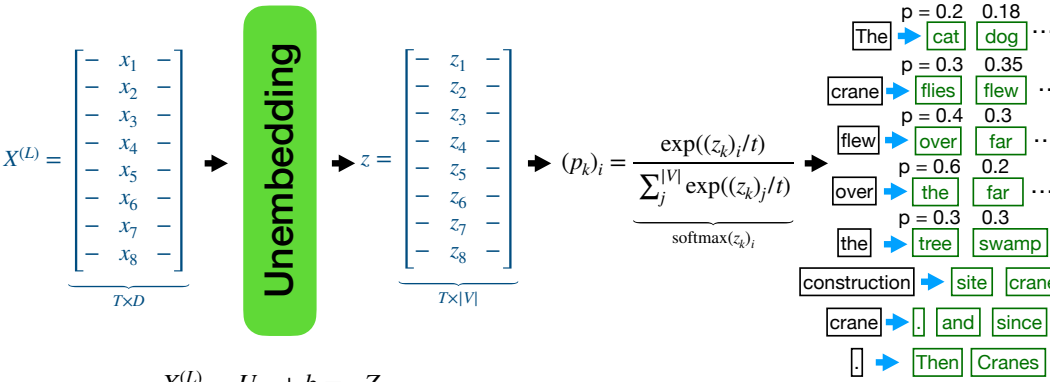

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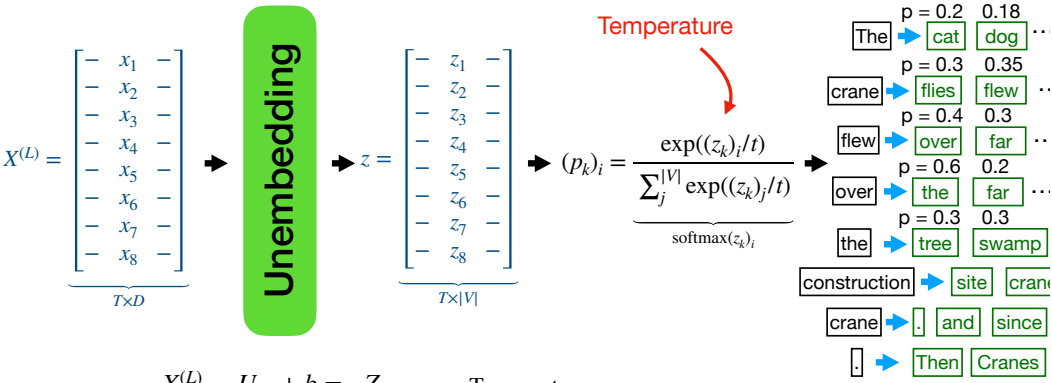


$$\underbrace{X^{(L)}}_{T\times D} \cdot \underbrace{U}_{D\times|V|} + b = \underbrace{Z}_{T\times|V|}$$

Temperature:

- t < 1: logits become larger relative to each other → sharper distribution → more deterministic.
- t > 1: logits shrink → softer distribution → more random.

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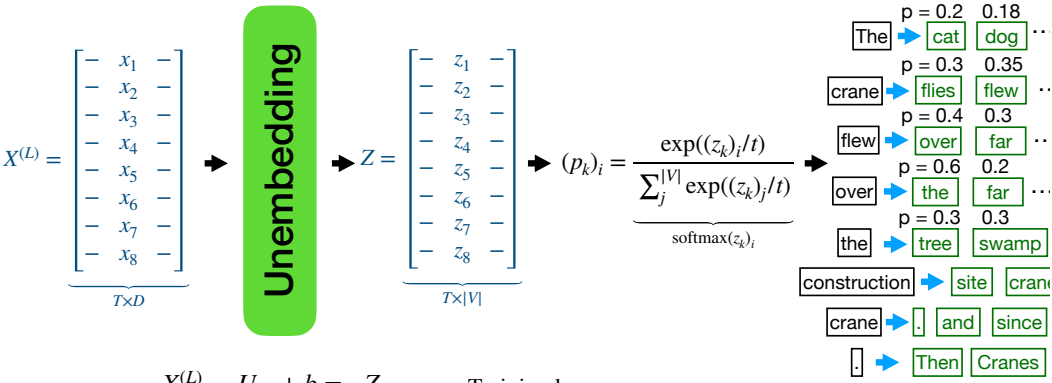


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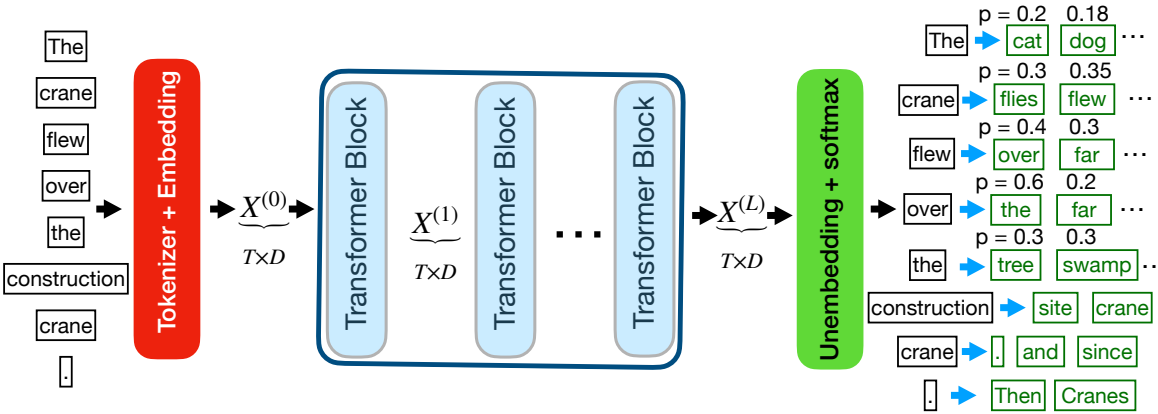
Training loss:

$$L(\theta) = -\sum_{t=1}^{7} \log p_\theta(x_{t+1} \mid x_{1:t}) = -\sum_{t=1}^{7} \log(p_k)_{x_{k+1}}$$

# Summary: Decoder Transformer



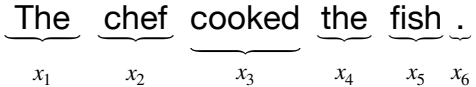

Tokenizer + Embedding : $\Sigma^* \to \mathbb{R}^{T \times D}$ string to sequence embedding

Transformer Block : $\mathbb{R}^{T \times D} \to \mathbb{R}^{T \times D}$ sequence embeddings

Unembedding + Softmax : $\mathbb{R}^{T \times D} \to \mathbb{R}^{T \times |V|}$ probability over token vocabulary

# Position Embeddings

# Position embedding: Absolute (learned) embedding




$$o_6 = \sum_{i=1}^{6} \frac{\exp\left(\langle q_6, k_i \rangle / \sqrt{d}\right)}{\sum_{j=1}^{6} \exp\left(\langle q_j, k_j \rangle / \sqrt{d}\right)} \cdot v_i$$

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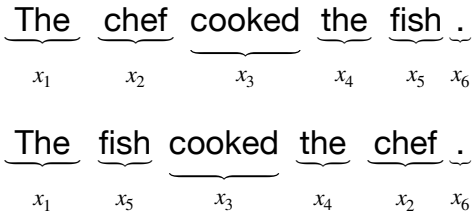


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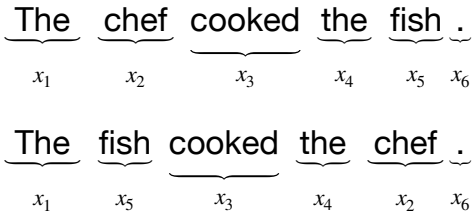


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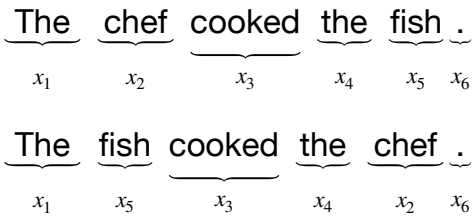


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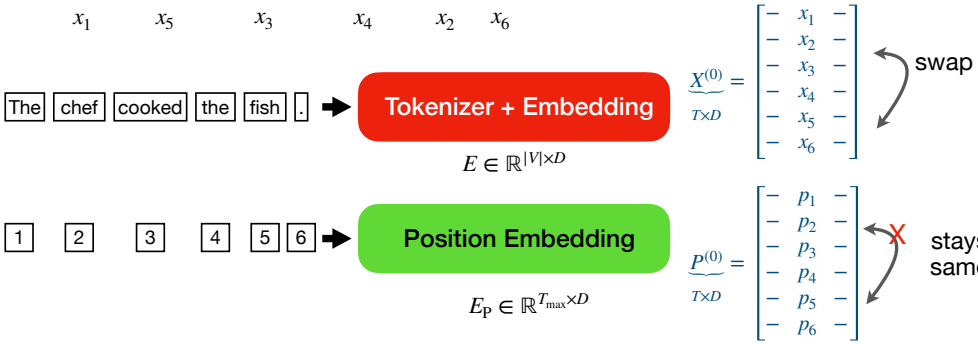

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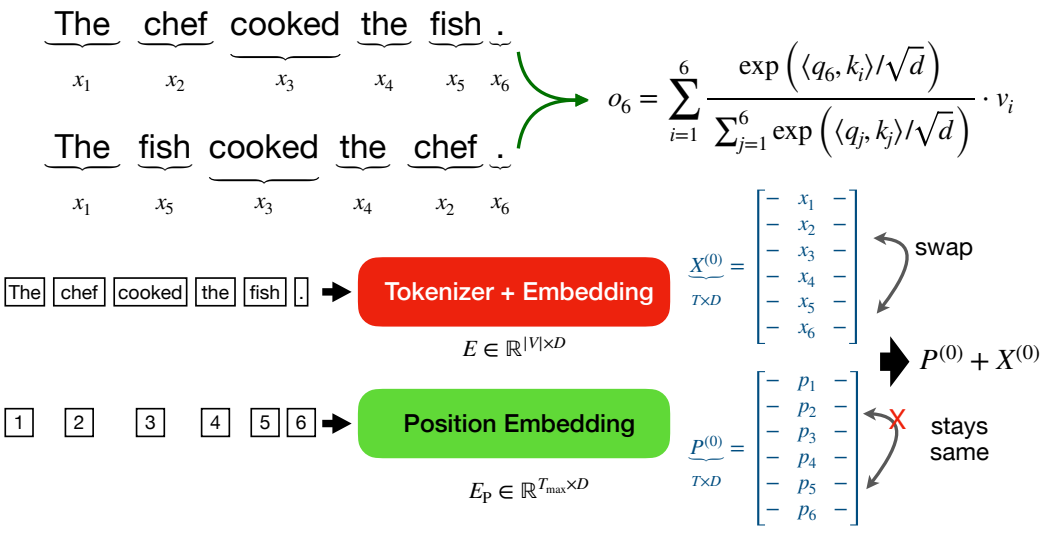

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**Want:** Inner product invariant to *absolute position* shifts

The constructed pos. embedding $f : \mathbb{R}^D \times \mathbb{N} \to \mathbb{R}^D$ has

$$\langle f(x_1, i), f(x_2, j) \rangle = g(x_1, x_2, i - j) \qquad \text{for some } g : \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{N} \to \mathbb{R}.$$

- The inner product depends only on:
  - token embeddings $x_1, x_2$
  - relative distance $i - j$

# RoPE: Rotary Position Embeddings **(Su et al., 2021)**

**Idea:** Rotation preserves inner products

For positions $p \in \{1, \ldots, T_{\max}\}$, if we have an orthogonal matrices $\{R_p\}_{p=1}^{T_{\max}}$, s.t.,

$$R_p^\top = R_{-p} \quad \text{and} \quad R_m^\top R_n = R_{n-m} \quad \Longrightarrow \quad \langle R_m x_1, R_n x_2 \rangle = \langle x_1, R_{n-m} x_2 \rangle.$$

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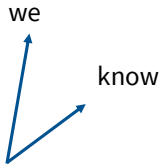
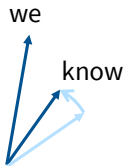
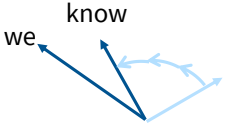



Position independent embedding




Embedding
“we know that”

Rotate we by ‘0 positions’
know by ‘1 positions’




Embedding
“of course we know”

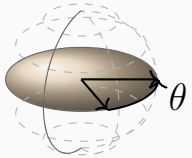
Rotate we by ‘2 positions’
Rotate know by ‘3 positions’

From CS336: Lecture 3 on Architectures

# RoPE: A suitable family of rotations in $\mathbb{R}^D$

## Rotation in 2D subspaces ("Givens rotation")

$$G(i, i+1, \theta) = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & I_{D-i-1} \end{pmatrix}$$



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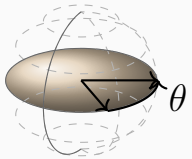


## RoPE

Let $\theta_i = 10\,000^{-2i/D}$ (high $i$ rotate slower)

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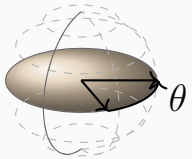


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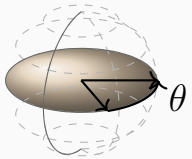


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- Can be viewed as sampled Fourier phases of a Dirac delta (point mass) at position $p$.

# RoPE: Where it is used?

## Where it is used?

For hidden states $X \in \mathbb{R}^{T \times D}$, first compute

$$Q = XW_Q, \qquad K = XW_K, \qquad V = XW_V.$$

Then, in each attention head and at each position $t$,

$$\widetilde{q}_t = R_{\theta,d_h}(t)\, q_t, \qquad \widetilde{k}_t = R_{\theta,d_h}(t)\, k_t, \qquad \widetilde{v}_t = v_t.$$

## Effect on attention scores

$$a_{mn} = \frac{\widetilde{q}_m^\top \widetilde{k}_n}{\sqrt{d_h}} = \frac{q_m^\top R_{\theta,d_h}(n-m)\, k_n}{\sqrt{d_h}}.$$

- RoPE is applied **after the** $Q, K, V$ **projections**, separately in **each head**.
- **High-frequency** components vary quickly with position $\Rightarrow$ capture **local** information.
- **Low-frequency** components vary slowly with position $\Rightarrow$ capture **global / long-range**.

# Part 3: Architecture Variations, Inference, Post-Training

# Part 3a) Architecture Variations

# Normalization / Stability

- Representations can grow across many Transformer blocks.

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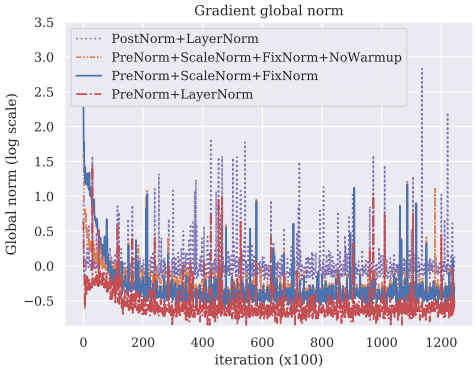
**Main role in Transformers** Keep each sublayer input well-conditioned before attention / FFN computations.



# Pre-norm vs post-norm

## Post-norm (original Transformer)

$$x_{l+1} = \mathrm{Norm}\big(x_l + F(x_l)\big)$$

- *After* the addition and MHA (or FFN).
- Harder to optimize in very deep Transformers.

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## Pre-norm (modern LLMs)

$$x_{l+1} = x_l + F(\mathrm{Norm}(x_l))$$

- *Before* MHA (or FFN).
- Cleaner residual path for gradients.
- More stable for deep, large-scale training.

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## Pre-norm (modern LLMs)

$$x_{l+1} = x_l + F(\mathrm{Norm}(x_l))$$

- *Before* MHA (or FFN).
- Cleaner residual path for gradients.
- More stable for deep, large-scale training.

Most modern LLMs use **pre-norm** because it trains more reliably at scale.

# LayerNorm vs RMSNorm

## LayerNorm

Normalizes both mean and variance across features:

$$\mathrm{LayerNorm}(x) = \gamma \odot \frac{x - \mathrm{E}[x]}{\sqrt{\mathrm{Var}[x] + \varepsilon}} + \beta$$

- centers and rescales + learned shift $\beta$,

**Typical examples (LayerNorm):**
Original Transformer, BERT, GPT-2/3, OPT

## RMSNorm

Normalizes only the root-mean-square:

$$\mathrm{RMSNorm}(x) = \gamma \odot \frac{x}{\sqrt{\frac{1}{d}\sum_{i=1}^{d} x_i^2 + \varepsilon}}$$

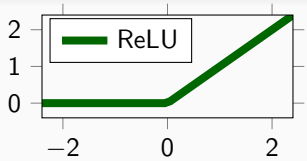
- does *not* subtract the mean,
- simpler and cheaper, widely used in modern LLMs.

**Typical examples (RMSNorm):**
T5, PaLM, LLaMA-family, Gemma

# Common activations: ReLU $\rightarrow$ GeLU

## ReLU (Rectified Linear Unit)

$$\mathrm{FFN}(x) = \max\{0, xW_1\}W_2$$

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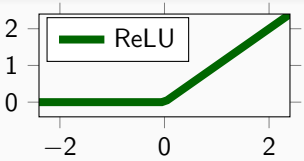


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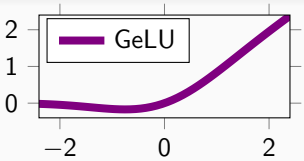


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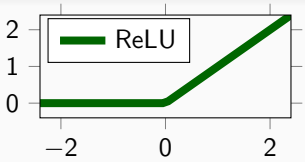


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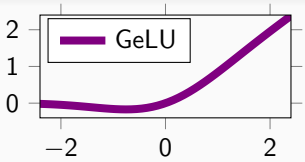


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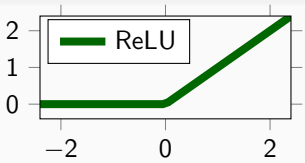


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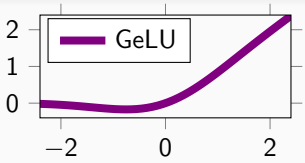


**Models:** GPT1/2/3, GPTJ, BERT, ...

- **Smoother non-linearity:** no non-smooth kink at 0.
- **Soft-gating:** preserves information of small negative activations (instead of setting them 0).

# Gated activations (Gated Linear Units, *GLU)

**Idea:** Separate "gating" from the linear transform

$$\max\{0, xW_1\} \quad \rightarrow \quad \max\{0, \underbrace{xW_1}_{\text{map for gating}}\} \otimes \underbrace{(xV)}_{\text{linear map on } x}$$

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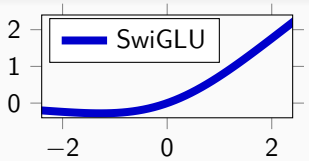
## SwiGLU (Swish Gated Linear Unit)

$\mathrm{Swish}(x) = x\,\mathrm{sigmoid}(x)$, where $\mathrm{sigmoid}(x) = 1/(1+e^{-x})$

$$\mathrm{FFN}_{\mathrm{SwiGLU}}(x) = (\mathrm{Swish}(xW_1) \otimes (xV))\, W_2$$




**Models:** LLaMA, PaLM, Mistral, ...

# Takeaways: Modern Transformer Decoder Block

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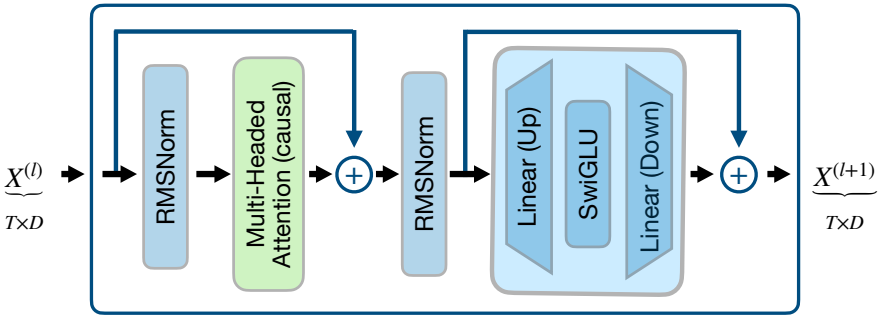
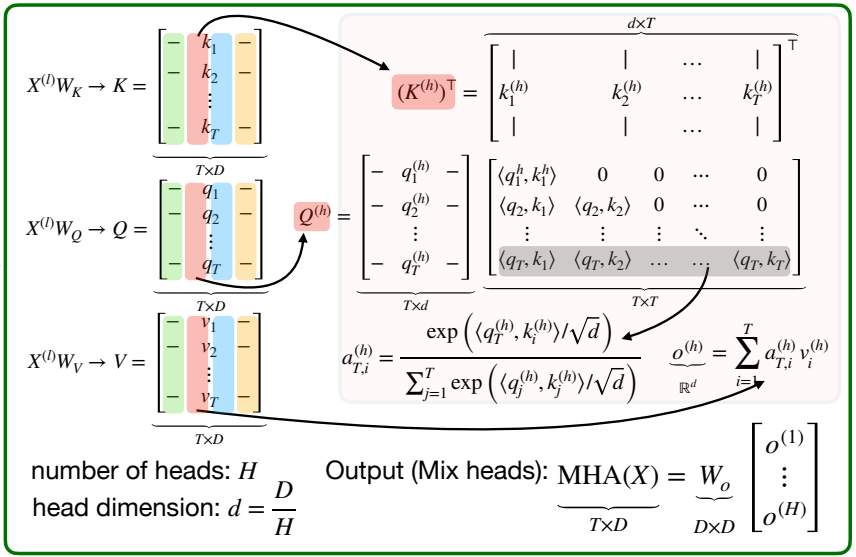
- **Parameters:** Biases ⇒ **Remove Biases**
  - Bias terms ($+b$) are dropped to save memory and increase throughput.
- **Positional Encoding:** Absolute (Input) ⇒ **RoPE** in every head of MHA
  - Injected directly *inside* MHA to model relative distances ⇒ improving long-context modelling.

# Decoder transformer



# Part 3b) Inference: Prefill vs Decode

# Recall: Multi-Headed Attention (MHA) with causal mask



# Metrics for Inference

## Metrics

- *Time-to-first-token (TTFT):* How long user waits before ay generation appears
- *Latency (seconds/token):* How fast tokens appear for user
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***Why important?***

- Providers of closed / open models must solve this problem
- Influences architecture

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**Conclusion:** compute-limited in BF16 if $B > 400$.

# MLP Arithmetic Summary

## Prefill ($T = S$)

$$\text{FLOPs} \approx 6\,B\,S\,D\,F$$

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## Takeaway

For $BT \ll D, F$,

$$\text{Intensity} = \mathcal{O}(BT).$$

So MLP layers benefit strongly from batching:

Prefill: increase $BT$, Generation: increase $B$.

# MHA Arithmetic (FlashAttn) Summary

$$\text{Intensity} = \frac{ST}{S+T}. \qquad \Rightarrow \qquad \text{Prefill: } \text{Intensity} = \frac{S}{2}, \qquad \text{Generation: } \text{Intensity} < 1$$

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## Takeaway

Batching $B$ improves throughput, *but not* the intensity (each request reads its own KV cache.)

# Transformer Block: Prefill vs Generation

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Input: many prompt tokens at once

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## Main takeaway

Prefill : TTFT vs throughput tradeoff  Generation: Intensity limits latency, throughput with $B$.

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Prefill: compute-rich prompt processing  Decode: memory-bound token loop

Batching improves throughput, but serving must balance TTFT and per-token latency.

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## ... but what to do about latency in generation?

# Grouped-query attention **(Ainslie et al., 2023)**

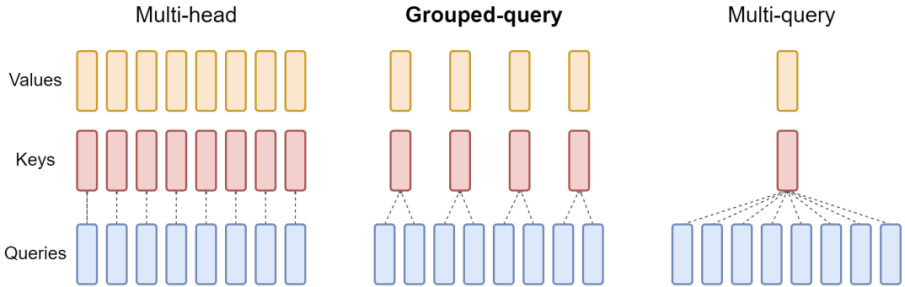


Figure 2: Overview of grouped-query method. Multi-head attention has H query, key, and value heads. Multi-query attention shares single key and value heads across all query heads. Grouped-query attention instead shares single key and value heads for each *group* of query heads, interpolating between multi-head and multi-query attention.

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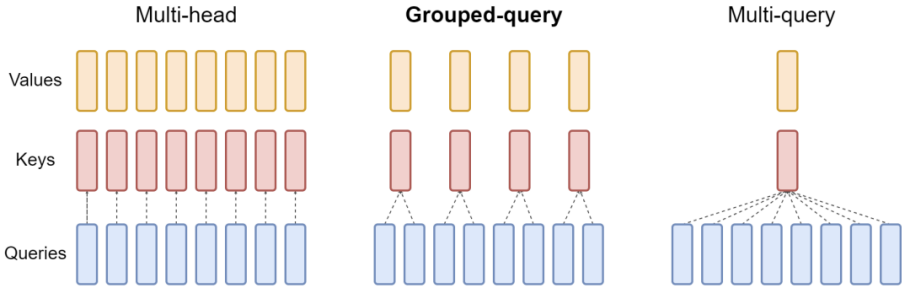


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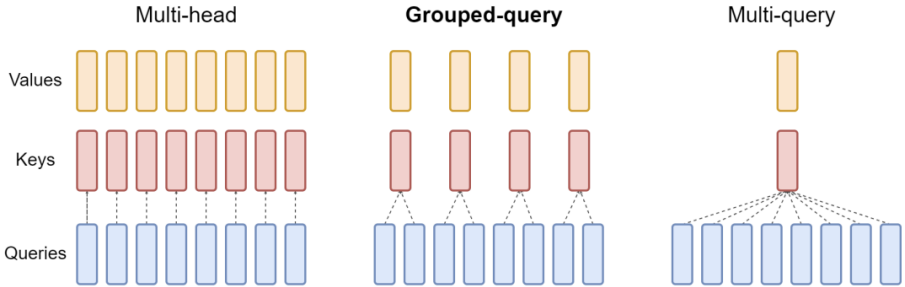


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**Why?**

Smaller KV cache and less memory traffic during generation.

# Why GQA Helps Inference

- **MHA:** $H_Q = H_K$
  - largest KV cache
  - best flexibility
- **MQA:** $H_K = 1$
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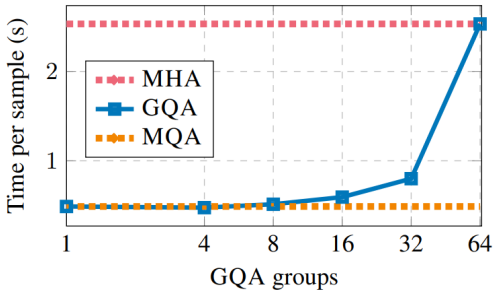

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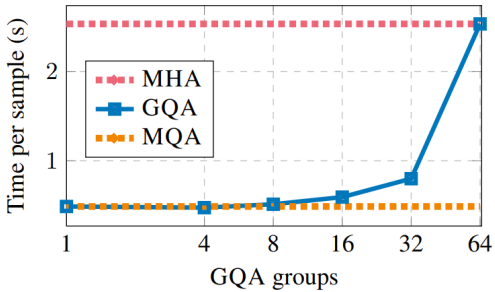


## Takeaway

GQA keeps many query heads, but few keys/values heads ⇒ reduce the memory cost + operations.

# GQA Arithmetic Summary

$$\text{Intensity}: \quad \frac{ST}{T+S} \Rightarrow \frac{ST}{T+S/G}$$

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## Takeaway

Prefill: still good, Generation: latency and memory lower than MHA .. and same FLOPS!

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- **Final takeaway:** LLM inference is not only a "systems" problem, but also an architecture modeling problem!

# Distribution of language ⇒ task-oriented model

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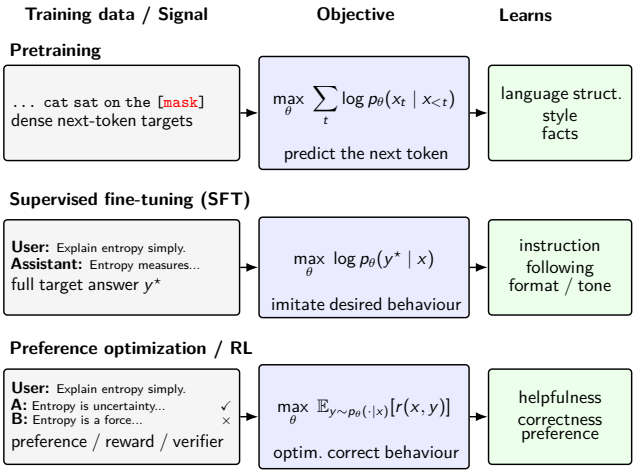

# Part 3c) Post-training: SFT, IT, RL

# Instruction tuning: objective and optimization

## Data and model

$$\mathcal{D} = \{(x^{(i)}, y^{(i)})\}_{i=1}^{N}$$

with instructions $x^{(i)}$ and desired responses $y^{(i)}$.
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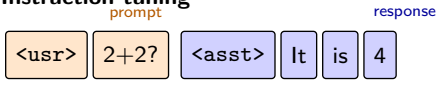
**Pretraining**


loss on all next-token predictions

**Instruction tuning**




prompt tokens: context only
assistant tokens: optimized

Feed the concatenated sequence $[x_i; y_i]$ into the causal LM, but mask out prompt tokens in the loss.

# PEFT: adapt a large model with a small trainable update

## Low-Rank Adaptation (LoRA)

Freeze the pretrained matrix $W_0$ and learn only a low-rank update:

$$W = W_0 + \Delta W, \qquad \Delta W = BA, \qquad r \ll d.$$

So instead of training all of $W$, we train only the small factors $A$ and $B$.

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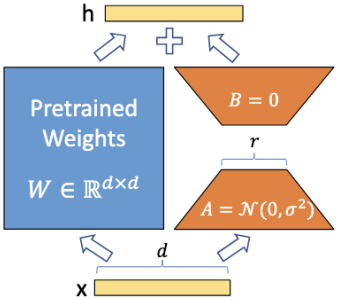


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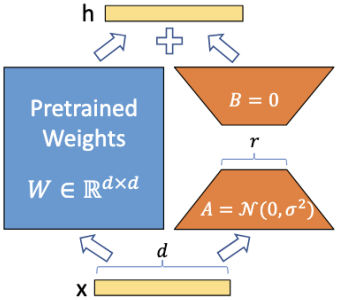


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## Other PEFT

- **Adapters:** bottleneck modules
- **QLORA:** Quantized LoRA

# FLAN: Instruction Tuning ⇒ Zero-shot Learning **(Wei et al., 2021)**

## What FLAN showed

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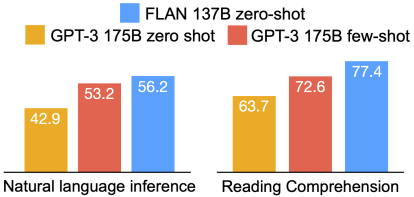
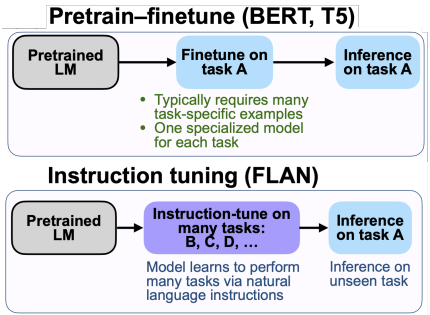



Performance on unseen tasks



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## Takeaway

- SFT is good at **extracting present knowledge**
- It is less reliable for **adding new capabilities**

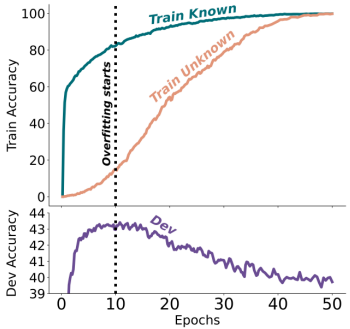
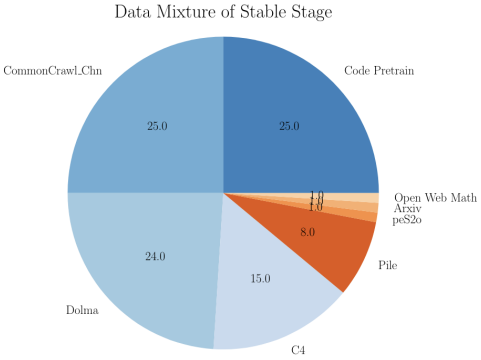
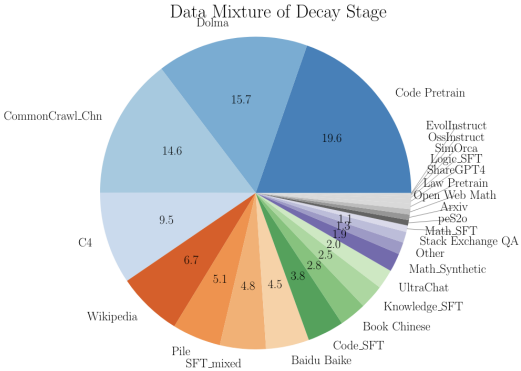



Figure 1: Train and development accuracies as a function of the fine-tuning duration, when fine-tuning on 50% `Known` and 50% `Unknown` examples. `Unknown` examples are fitted substantially slower than `Known`. The best development performance is obtained when the LLM fits the majority of the `Known` training examples but only few of the `Unknown` ones. From this point, fitting `Unknown` examples reduces the performance.

# Midtraining / Two-phase training

- train on a **mixture** of web-scale text and instruction data
- **injects instruct behaviour** without leaving the pretraining regime entirely
- preserves **general knowledge and language modelling ability**
- reduces **catastrophic forgetting:** the model loosing previously learned capabilities.



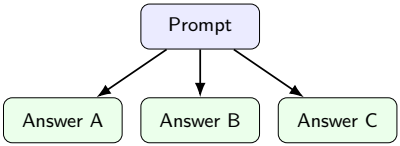



# When instruction tuning is not sufficient

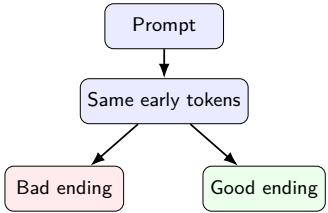
## Instruction tuning is limited

- **There is no single "correct" path**
- **Quality is judged by outcome**
- **The important signal appears only after generation**
- **Different mistakes have different severity**
- **The model may exploit superficial patterns**
- **Preferences are often relative, not absolute**

*In short: instruction tuning learns from fixed targets, but many post-training signals are comparative or outcome-based.*

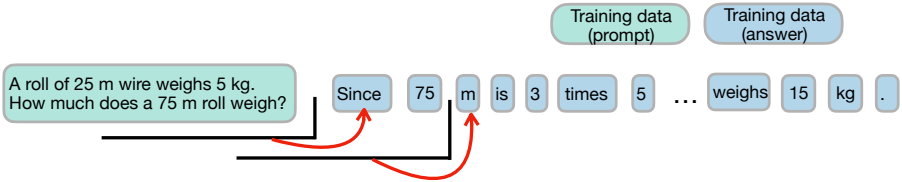



**Many plausible responses:** instruction tuning reinforces one recorded target, even when several answers are acceptable.




**Outcome visible late:** two responses may look similar at first, but only the full completion reveals whether it is correct, safe, or useful.

# SFT $\Rightarrow$ RL




Compute loss of $p_\theta(\text{"m"} \mid \text{"Since"}, \text{"75"}, \text{prompt}) \Rightarrow$ gradient update on $\theta$ to increase the prob. of observed training distribution

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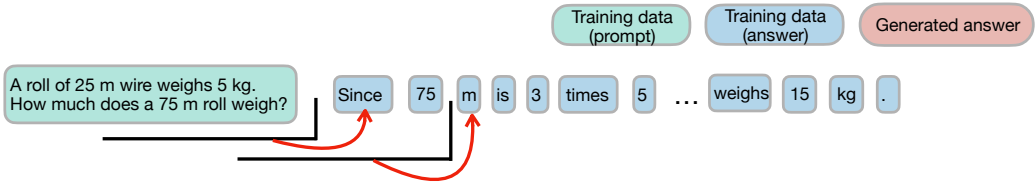


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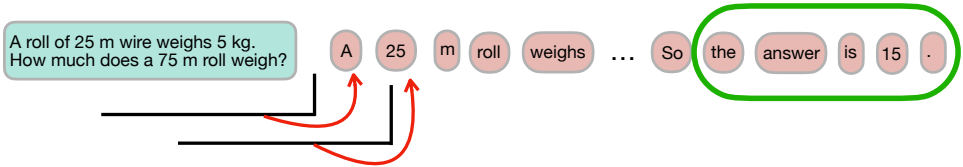


Generate from "m" $\sim \pi_\theta(\cdot \mid \text{"A"}, \text{"25"}, \text{prompt}) \Rightarrow$ check correctness $\Rightarrow$ push probability of path up

# SFT vs RL

## Supervised Fine-Tuning (SFT)

$$\max_{\theta} \mathbb{E}_{(x,y^\star)\sim\mathcal{D}}\big[\log \pi_\theta(y^\star \mid x)\big]$$

- **Target:** *"do this"*
- Full desired output $y^\star$ is given
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## Challenges of using rewards

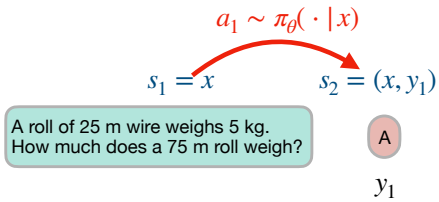
- **Scalar vs structured:** a label specifies the desired output; a reward is one number.
- **Credit assignment:** the reward arrives after the whole response, so it is unclear which part of reasoning are responsible for it.

# RL formulation for an LLM

action $a_t$ : sampled token $y_t \sim p_\theta(\cdot \mid x, y_{<t})$

state $s_t$ : prompt + partial completion $(x, y_{<t})$

policy $\pi_\theta$ : LLM next tok. prediction



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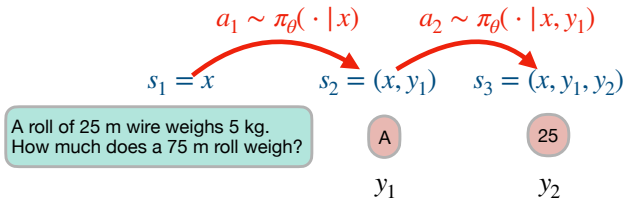

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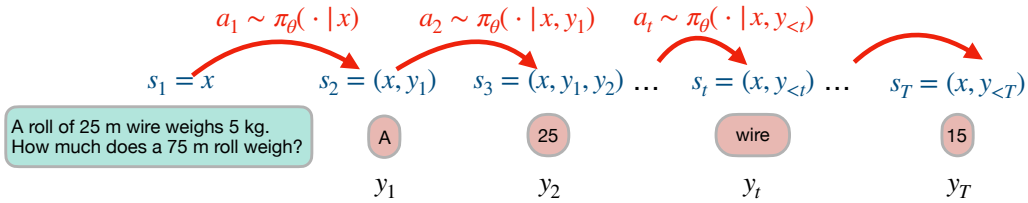


rollout $y_{\leq T}$ : fully AR sampled answer

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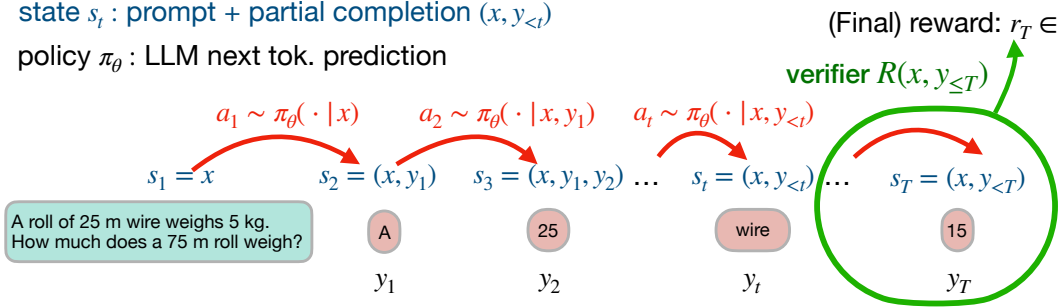


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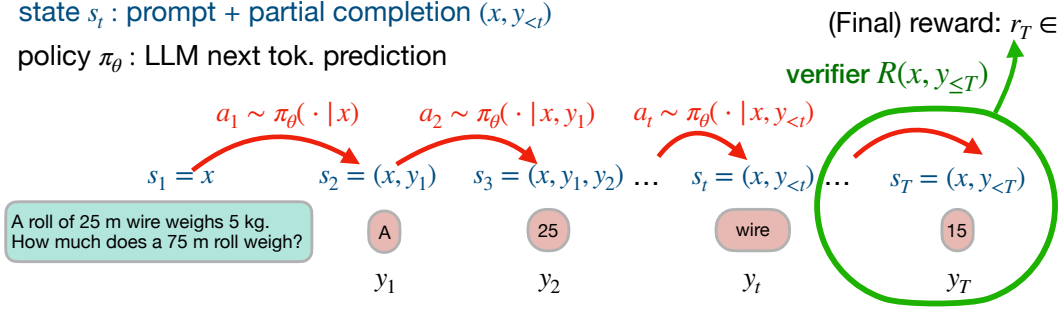


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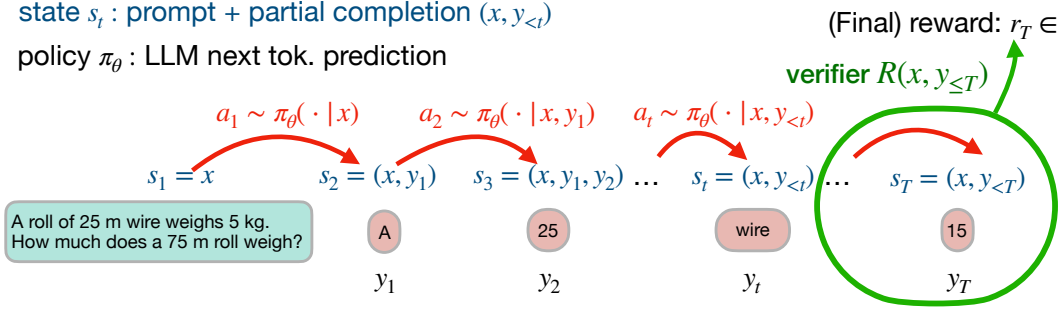


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Value function: $V^\pi(s_t) = \mathbb{E}_{a \sim \pi_\theta(\cdot \mid s_t)} \left[ G_t \mid s_t \right]$ $\Longleftarrow$ Expected future reward under $\pi_\theta$ from $s_t$

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**Objective:**

$$\max_{\theta} J(\theta) := \mathbb{E}_{x\sim D, y\sim\pi_\theta(\cdot|x)}\left[R(x,y)\right]$$

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**REINFORCE:**

- sample prompt $x \sim D$
- answer $y \sim \pi_\theta(\,\cdot\,|\,x)$
- score reward $R(x,y) \in \mathbb{R}^d$

$$\theta^{(t+1)} = \theta^{(t)} + \alpha\, R(x,y) \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(y_t \mid x, y_{<t})$$

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  - Evolution of language models into LLMs (motivations, challenges, ...)
  - GPT2-like architecture (tokenizer, transformer, MHA)
  - Further architecture variations (RoPE, SwiGLU, pre-Norm)
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- **Recommended Sources**:
  - CS336 Language Modeling from Scratch
  - Karpathy's nanoGPT + microGPT, link
  - CS224N NLP with Deep Learning
  - Blog "A History of Large Language Models" by Gregory Gundersen